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ORIGINAL ARTICLE

A binding number condition for graphs to be (a, b, k)-critical graphs ☆

Sizhong Zhou [a,*], **Jiashang Jiang** [a], **Lan Xu** [b]

[a] *School of Mathematics and Physics, Jiangsu University of Science and Technology, Mengxi Road 2, Zhenjiang, Jiangsu 212003, People's Republic of China*
[b] *Department of Mathematics, Changji University, Changji, Xinjiang 831100, People's Republic of China*

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Abstract Let a and b be two even integers with $2 \leqslant a < b$, and let k be a nonnegative integer. Let G be a graph of order n. Its binding number $bind(G)$ is defined as follows,

$$bind(G) = min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G),\ N_G(X) \neq V(G)\right\}.$$

In this paper, it is proved that G is an (a, b, k)-critical graph if $bind(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+3}$ and $n \geqslant \frac{(a+b)(a+b-3)}{b} + \frac{bk}{b-1}$. Furthermore, it is shown that the result in this paper is best possible in some sense.

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* Corresponding author.
E-mail address: zsz_cumt@163.com (S. Zhou).

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1. Introduction

The graphs considered here will be finite undirected graphs without loops or multiple edges. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G, and by $N_G(x)$ the set of vertices adjacent to x in G. The minimum vertex degree of G is denoted by $\delta(G)$. For $S \subseteq V(G)$, $N_G(S) = \cup_{x\in S} N_G(x)$. For a nonempty subset S of $V(G)$ we denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G)\backslash S]$ for a proper subset S of $V(G)$. We say that S is independent if $N_G(S) \cap S = \emptyset$. The binding number $bind(G)$ of G is defined by

$$bind(G) = min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

Let a and b be integers with $0 \leqslant a \leqslant b$. An $[a,b]$-factor of a graph G is defined as a spanning subgraph F of G such that $a \leqslant d_F(x) \leqslant b$ for every vertex x of G (where of course d_F denotes the degree in F). And if $a = b = r$, then an $[a,b]$-factor is called an r-factor. A graph G is called an (a,b,k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a,b]$-factor. If G is an (a,b,k)-critical graph, then we also say that G is (a,b,k)-critical. If $a = b = r$, then an (a,b,k)-critical graph is simply called an (r,k)-critical graph. In particular, a $(1,k)$-critical graph is simply called a k-critical graph.

Many authors [1,2] investigated the graphs factors. Liu and Yu [6] studied the characterization of (r,n)-critical graphs. Liu and Wang [5] gave the characterization of (a,b,k)-critical graphs with $a < b$. Li [3,4] showed three sufficient conditions for graphs to be (a,b,k)-critical graphs. Zhou [9,11,10] obtained some sufficient conditions for graphs to be (a,b,k)-critical graphs. Liu and Liu [7] gave a binding number and minimum degree condition for a graph to be an (a,b,k)-critical graph. The following result on (a,b,k)-critical graphs was proved by Zhou and Jiang in [11].

Theorem 1 (*11*). *Let a, b and k be nonnegative integers with $1 \leqslant a < b$. Let G be a graph of order n with $n \geqslant \frac{(a+b-1)(a+b-2)}{b} + \frac{bk}{b-1}$, and suppose that*

$$bind(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+2}.$$

Then G is an (a,b,k)-critical graph.

Zhou and Jiang [11] also showed that the condition $bind(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+2}$ in Theorem 1 can not be replaced by $bind(G) \geqslant \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+2}$. For the proof of optimality (in this sense), they considered the case when $a + b + k$ is odd and $n = \frac{(a+b)(a+b-2)+(a+2b-1)k}{b}$ is an integer. Then they constructed a non (a,b,k)-critical graph G with $bind(G) = \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+2}$. It is easy to see that in this case, either a and b are both odd, or a is even and b is odd. Thus, the question is:

Is the condition $bind(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+2}$ optimal in the other cases? (i.e. when (a and b are both even) or (a is odd and b is even)).

In this paper, we study this question when the integers a and b are both even. In this case, we improve our previous result and obtain the following theorem.

Theorem 2. *Let a and b be two even integers with $2 \leqslant a < b$, and let k be a nonnegative integer. Let G be a graph of order n with $n \geqslant \frac{(a+b)(a+b-3)}{b} + \frac{bk}{b-1}$, and suppose that*

$$bind(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+3}.$$

Then G is an (a,b,k)-critical graph.

If $k = 0$ in Theorem 2, then we get the following corollary.

Corollary 1. *Let a and b be two even integers with $2 \leqslant a < b$. Let G be a graph of order n with $n \geqslant \frac{(a+b)(a+b-3)}{b}$, and suppose that*

$$bind(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)+3}.$$

Then G has an $[a,b]$-factor.

2. Preliminary lemmas

Let a and b be two positive integers with $a < b$, and let G be a graph. For any $S \subseteq V(G)$, define

$$d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$$

and

$$\delta_G(S,T) = b|S| + d_{G-S}(T) - a|T|,$$

where $T = \{x\colon x \in V(G) \setminus S,\ d_{G-S}(x) \leqslant a-1\}$. In the following, we define

$$h = \min\{d_{G-S}(x) : x \in T\}.$$

Obviously, $0 \leqslant h \leqslant a-1$.

Liu and Wang [5] proved the following result which is applied in the proof of the theorems.

Lemma 2.1 [5]. *Let a, b and k be nonnegative integers with $1 \leqslant a < b$, and let G be a graph of order $n \geqslant a + k + 1$. Then G is (a,b,k)-critical if and only if for any $S \subseteq V(G)$ with $|S| \geqslant k$*

$$\delta_G(S, T) \geqslant bk,$$

where $T = \{x : x \in V(G) \setminus S,\ d_{G-S}(x) \leqslant a-1\}$.

Lemma 2.2. [8]*Let c be a positive real, and let G be a graph of order n with bind(G)* $> c$. *Then* $\delta(G) > n - \frac{n-1}{c}$.

Lemma 2.3. *Let a and b be two even integers with* $2 \leqslant a < b$, *and let k be a nonnegative integer. Let G be a graph of order n. If* $\delta_G(S,T) \leqslant bk-1$ *for some* $S \subseteq V(G)$, *then* $|S| \leqslant \frac{(a-h)n+bk-2}{a+b-h}$.

Proof 1. By the definition of h and the condition of Lemma 2.3, we have

$$\begin{aligned} bk-1 &\geqslant \delta_G(S, T) = b|S| + d_{G-S}(T) - a|T| \geqslant b|S| + h|T| - a|T| \\ &= b|S| - (a-h)|T|, \end{aligned}$$

that is,

$$b|S| - (a-h)|T| - bk \leqslant -1. \tag{1}$$

Case 1. h is even.

In this case, the left-hand side of (1) is even, thus

$$b|S| - (a-h)|T| - bk \leqslant -2. \tag{2}$$

According to (2), $0 \leqslant h \leqslant a-1$ and $|S| + |T| \leqslant n$, we obtain

$$\begin{aligned} bk-2 &\geqslant b|S| - (a-h)|T| \geqslant b|S| - (a-h)(n-|S|) \\ &= (a+b-h)|S| - (a-h)n, \end{aligned}$$

which implies

$$|S| \leqslant \frac{(a-h)n+bk-2}{a+b-h}.$$

Case 2. h is odd.

Subcase 2.1. There exists $x \in T$ such that $d_{G-S}(x) \geqslant h+1$. In this case, we get

$$d_{G-S}(T) \geqslant h|T| + 1. \tag{3}$$

In terms of (3), $\delta_G(S,T) \leqslant bk-1, 0 \leqslant h \leqslant a-1$ and $|S| + |T| \leqslant n$, we have

$$\begin{aligned} bk-1 &\geqslant \delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \geqslant b|S| + h|T| + 1 - a|T| \\ &= b|S| - (a-h)|T| + 1 \geqslant b|S| - (a-h)(n-|S|) + 1 \\ &= (a+b-h)|S| - (a-h)n + 1, \end{aligned}$$

that is,

$$|S| \leqslant \frac{(a-h)n + bk - 2}{a+b-h}.$$

Subcase 2.2. $V(G)\backslash(S \cup T) \neq \emptyset$.
In this case, we obtain

$$|S| + |T| \leqslant n - 1. \tag{4}$$

From (1) and (4) and $0 \leqslant h \leqslant a-1$, we have

$$\begin{aligned} bk-1 &\geqslant b|S| - (a-h)|T| \geqslant b|S| - (a-h)(n-1-|S|) \\ &= (a+b-h)|S| - (a-h)n + (a-h) \\ &\geqslant (a+b-h)|S| - (a-h)n + 1, \end{aligned}$$

which implies

$$|S| \leqslant \frac{(a-h)n + bk - 2}{a+b-h}.$$

Subcase 2.3. $V(G)\backslash(S \cup T) = \emptyset$ and $d_{G-S}(x) = h$ for each $x \in T$.In this case, $d_{G[T]}(x) = h$ for each $x \in T$. Since h is odd, $|T|$ is even. Thus, the left-hand side of (1) is even. Therefore, we obtain

$$bk - 2 \geqslant b|S| - (a-h)|T|.$$

Combining this with $|S| + |T| = n$, we have

$$bk - 2 \geqslant b|S| - (a-h)(n-|S|) = (a+b-h)|S| - (a-h)n,$$

that is,

$$|S| \leqslant \frac{(a-h)n + bk - 2}{a+b-h}.$$

This completes the proof of Lemma 2.3. □

3. The proof of Theorem 2

Proof 2. Let G be a graph satisfying the hypothesis of Theorem 2. We prove the theorem by contradiction. Suppose that G is not an (a,b,k)-critical graph. Then by Lemma 2.1, there exists a subset S of $V(G)$ with $|S| \geqslant k$ such that

$$\delta_G(S, T) \leqslant bk - 1, \tag{5}$$

where $T = \{x{:}x \in V(G)\backslash S,\ d_{G-S}(x) \leqslant a - 1\}$. Clearly, $T \neq \emptyset$ by (5). Let h be as in Section 2, and $0 \leqslant h \leqslant a - 1$.

We shall consider various cases by the value of h and derive contradictions.

Case 1. $h = 0$. Let $X = \{x{:}\ x \in T, d_{G-S}(x) = 0\}$. Then $X \neq \emptyset$ and $N_G(V(G)\backslash S) \cap X = \emptyset$. According to the definition of $bind(G)$ and the condition of Theorem 2, we have

$$\frac{(a+b-1)(n-1)}{bn-(a+b)-bk+3} < bind(G) \leqslant \frac{|N_G(V(G) \setminus S)|}{|V(G) \setminus S|} \leqslant \frac{n-|X|}{n-|S|}. \tag{6}$$

Now we prove the following claim. □

Claim 1. $bn - (a + b) - bk + 2 > n - 1.$

Proof 3. According to $n \geqslant \frac{(a+b)(a+b-3)}{b} + \frac{bk}{b-1}$ and $2 \leqslant a < b$, we have

$$\begin{aligned}
&b(bn-(a+b)-bk+2-(n-1)) = b((b-1)n-(a+b)-bk+3)\\
&\geqslant b(b-1)\left(\frac{(a+b)(a+b-3)}{b}+\frac{bk}{b-1}\right)-b(a+b)-b^2k+3b\\
&=(b-1)(a+b)(a+b-3)+b^2k-b(a+b)-b^2k+3b\\
&=(b-1)(a+b)(a+b-3)-b(a+b-3)\\
&=(a+b-3)((b-1)(a+b)-b)>(a+b-3)((a+b)-b)=a(a+b-3)>0.
\end{aligned}$$

Thus, we obtain

$$bn-(a+b)-bk+2>n-1.$$

This completes the proof of Claim 1. □

In terms of (6), $n \geqslant \frac{(a+b)(a+b-3)}{b} + \frac{bk}{b-1}$, $|X| \geqslant 1$ and Claim 1, we obtain

$$\begin{aligned}
&(a+b-1)(n-1)|S| > (a+b-1)(n-1)n-(bn-(a+b)-bk+3)n\\
&+(bn-(a+b)-bk+3)|X| = (a-1)(n-1)n+(a-2)n\\
&+(bk-1)n+(bn-(a+b)-bk+3)|X| \geqslant (a-1)(n-1)n\\
&+(bk-1)n+(bn-(a+b)-bk+3)|X| = (a-1)(n-1)n\\
&+(bk-1)(n-1)+bk-1+(bn-(a+b)-bk+3)|X|\\
&\geqslant (a-1)(n-1)n+(bk-1)(n-1)+(bn-(a+b)-bk+2)|X|\\
&> (a-1)(n-1)n+(bk-1)(n-1)+(n-1)|X|,
\end{aligned}$$

which implies

$$|S| > \frac{(a-1)n + bk - 1 + |X|}{a+b-1}. \tag{7}$$

On the other hand, by (5) and $|S| + |T| \leqslant n$, we have

$$\begin{aligned} bk - 1 &\geqslant \delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \geqslant b|S| - (a-1)|T| - |X| \\ &\geqslant b|S| - (a-1)(n - |S|) - |X| = (a+b-1)|S| - (a-1)n - |X|, \end{aligned}$$

that is,

$$|S| \leqslant \frac{(a-1)n + bk - 1 + |X|}{a+b-1},$$

which contradicts (7).

Case 2. $1 \leqslant h \leqslant a-1$.

According to Lemma 2.2 and the hypothesis of Theorem 2, we have

$$\delta(G) > n - \frac{bn - (a+b) - bk + 3}{a+b-1} = \frac{(a-1)n + (a+b) + bk - 3}{a+b-1}. \tag{8}$$

We choose $x_1 \in T$ such that $d_{G-S}(x_1) = h$. Thus, we obtain

$$|S| + h = |S| + d_{G-S}(x_1) \geqslant d_G(x_1) \geqslant \delta(G).$$

Combining this with (8), we have

$$|S| \geqslant \delta(G) - h > \frac{(a-1)n + (a+b) + bk - 3}{a+b-1} - h. \tag{9}$$

Subcase 2.1. $h = 1$. From (9), we get

$$|S| > \frac{(a-1)n + (a+b) + bk - 3}{a+b-1} - 1 = \frac{(a-1)n + bk - 2}{a+b-1},$$

which contradicts Lemma 2.3.

Subcase 2.2. $2 \leqslant h \leqslant a-1$. In terms of Lemma 2.3 and (9), we obtain

$$\frac{(a-h)n + bk - 2}{a+b-h} + h > \frac{(a-1)n + (a+b) + bk - 3}{a+b-1}. \tag{10}$$

Set $f(h) = \frac{(a-h)n+bk-2}{a+b-h} + h$. Using $n \geqslant \frac{(a+b)(a+b-3)}{b} + \frac{bk}{b-1}$, we have

$$\begin{aligned} f'(h) &= \frac{(a+b-h)(-n) + (a-h)n + bk - 2}{(a+b-h)^2} + 1 = \frac{-bn + bk - 2}{(a+b-h)^2} + 1 \\ &\leqslant \frac{-bn + bk - 2}{(a+b-2)^2} + 1 \leqslant \frac{-((a+b)(a+b-3) + bk) + bk - 2}{(a+b-2)^2} + 1 = -\frac{1}{a+b-2} < 0. \end{aligned}$$

Thus, we get

$$f(h) \leqslant f(2). \tag{11}$$

Claim 2. $\frac{(a-2)n+bk-2}{a+b-2} + 2 \leqslant \frac{(a-1)n+(a+b)+bk-3}{a+b-1}$.

Proof 4. According to $n \geqslant \frac{(a+b)(a+b-3)}{b} + \frac{bk}{b-1}$ and $2 \leqslant a < b$, we have

$$\begin{aligned}
&(a+b-1)(a+b-2)\left(\frac{(a-1)n+(a+b)+bk-3}{a+b-1} - \frac{(a-2)n+bk-2}{a+b-2} - 2\right) \\
&\quad = (a+b-2)(a-1)n + (a+b-2)(a+b-3) + (a+b-2)bk \\
&\quad - (a+b-1)(a-2)n - (a+b-1)bk - 2(a+b-1)(a+b-3) \\
&\quad = bn - (a+b)(a+b-3) - bk \geqslant b\left(\frac{(a+b)(a+b-3)}{b} + \frac{bk}{b-1}\right) \\
&\quad - (a+b)(a+b-3) - bk = \frac{bk}{b-1} \geqslant 0.
\end{aligned}$$

Thus, we have

$$\frac{(a-2)n+bk-2}{a+b-2} + 2 \leqslant \frac{(a-1)n+(a+b)+bk-3}{a+b-1}.$$

This completes the proof of Claim 2. □

By Claim 2, we obtain

$$f(2) = \frac{(a-2)n+bk-2}{a+b-2} + 2 \leqslant \frac{(a-1)n+(a+b)+bk-3}{a+b-1}.$$

Combining this with (10) and (11), we get

$$\frac{(a-1)n+(a+b)+bk-3}{a+b-1} < f(h) \leqslant f(2) \leqslant \frac{(a-1)n+(a+b)+bk-3}{a+b-1}.$$

It is a contradiction.

From the argument above, we deduce the contradictions. Hence, G is an (a,b,k)-critical graph.

This completes the proof of Theorem 2. □

Remark 1. Let us show that the condition $bind(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+3}$ in Theorem 2 can not be replaced by $bind(G) \geqslant \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+3}$. Let a,b and k be three even integers such that $2 \leqslant a < b$ and $\frac{(a+b)(a+b-3)+(a+2b-1)k}{b}$ is an integer. We write $n = \frac{(a+b)(a+b-3)+(a+2b-1)k}{b}$, $l = \frac{a+b+k}{2} - 1$ and $m = n - 2l = n - (a+b+k-2) =$

$\frac{(a-1)(a+b-2)-2+(a+b-1)k}{b}$. Clearly, m,n,l are three positive integers. Let $G = K_m \bigvee lK_2$. Let $X = V(lK_2)$, then for any $x \in X$, $|N_G(X\backslash\{x\})| = n-1$. By the definition of $bind(G)$, $bind(G) = \frac{|N_G(X\backslash\{x\})|}{|X\backslash\{x\}|} = \frac{n-1}{2l-1} = \frac{n-1}{a+b+k-3} = \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+3}$. Let $S = V(K_m)$, $T = V(lK_2)$, then $|S| = m \geqslant k$, $|T| = 2l$. Thus, we obtain

$$\begin{aligned}\delta_G(S,T) &= b|S| - a|T| + d_{G-S}(T) = b|S| - a|T| + |T| = b|S| - (a-1)|T| \\ &= b\frac{(a-1)(a+b-2)-2+(a+b-1)k}{b} - (a-1)(a+b+k-2) \\ &= bk-2 < bk.\end{aligned}$$

By Lemma 2.1, G is not an (a,b,k)-critical graph. In the above sense, the result of Theorem 2 is best possible.

Remark 2. Zhou and Jiang [11] proved Theorem 1, and showed that the condition $bind(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+2}$ is sharp when either a and b are both odd, or a is even and b is odd. In this paper, we improve the binding number condition by $bind(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+3}$ when a and b are both even, and show that the condition in this case is sharp. Thus, we present the following problem:

Let a, b and k be three nonnegative integers such that $1 \leqslant a < b$, a is odd and b is even. Suppose that n is sufficiently large for a, b and k, and $bind(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+3}$. Then, whether a graph G of order n is (a,b,k)-critical or not?

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